INT. J. CONTROL, 1984, VOL. 40, NO. 1, 179–192

On the order of the minimal output representation of stochastic linear systems

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The relationship between the internal structure of a stochastic linear system in state-space form and the order of its output representation is investigated. Conditions for minimal and for maximal values of the output representation's order are derived.

1. Introduction

This paper is concerned with the correspondence between the order of a given stochastic linear system and that of its minimal output representation. From a practical viewpoint, this correspondence has a direct implication on the ability to determine the order of the system from the output alone and on the ability to design lower-order output filters and feedback controllers. In the stationary case, a minimal-order representation of the output may be obtained by factorizing the associated spectral density matrix (Wiener 1949, MacFarlane 1971), or by decomposing the Hankel matrix of output correlations (Akaike 1974, Faurre 1976, Baram 1981). In the univariable (single-input single-output) case, a transfer function is a minimal output representation under the assumption of minimum phase and analyticity at the origin (Stoica 1981). In the multivariable case, however, additional structural conditions are required. A condition for the minimality of a multivariable ARMA model was given by Hannan (1975) for the case where the model is of minimum phase and analytic at the origin and the input and the output are of the same dimensions.

In this paper we seek conditions on the internal structure of a multivariable system in state-space form for the equality of its order and the minimal order of its output representation. Our analysis is centred about the output correlation matrix, whose rank is equal to the order of the minimal output representation. The situation where the correlation matrix rank is smaller than the system's order is described as ' mode cancellation ', while the case of equality is described as ' mode retention '. (We refrain from using the term ' order reduction ' which is usually used to describe a design objective.) Conditions for mode cancellation and retention are obtained in terms of the input and the output structure and the internal (eigenvalue–eigenvector) structure of the system. A question which arises in connection to our analysis is whether cases of mode cancellation and retention are ' generic ', i.e. whether they are likely to occur in practice. While we make no attempt to answer this question in a precise sense (as we do not assign measures to the occurrence of different system structures), the examples are designed to provide intuitive answers to this question.

Received 20 August 1983.

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2. Preliminary discussion

We consider the asymptotically stable system

$$\mathbf{x}_{k+1}=F\mathbf{x}_k+G\mathbf{w}_k$$
$$\mathbf{y}_k=H\mathbf{x}_k+\mathbf{v}_k \tag{2.1}$$

where $\mathbf{x}_k \in R^p$, $\mathbf{w}_k \in R^r$, $\mathbf{v}_k$, $y_k \in R^q$ and where $\mathbf{w}_k$ and $\mathbf{v}_k$ are mutually uncorrelated white noise sequences. We assume that the system is observable and that H is of full rank. We denote the spectral decomposition of F by

$$F=U\Lambda U^{-1} \tag{2.2 a}$$

where

$$\Lambda=\operatorname{diag}\left[\Lambda_1 \ldots \Lambda_s \,\vdots\, \Lambda_{s+1} \ldots \Lambda_n\right] \tag{2.2 b}$$

$$\Lambda_i=\lambda_i I_{\sigma_i}, \quad \sigma_i>1, \quad \lambda_i \neq \lambda_j, \quad i, j=1, \ldots, s \tag{2.2 c}$$

$$\Lambda_i=\begin{bmatrix} \lambda_i & 1 \ldots & 0 \\ & \ddots & \vdots \\ & \ddots & 1 \\ 0 & & \lambda_i \end{bmatrix} \updownarrow \sigma_i, \quad \sigma_i \geqslant 1, \quad i=s+1, \ldots, n \tag{2.2 d}$$

and

$$\sum_{i=1}^{n} \sigma_i=p \tag{2.2 e}$$

Obviously, because the system (2.1) is assumed to be minimal, $\sigma_i \leqslant q$, $i \leqslant s$.

If there exist more than one Jordan block for $\lambda=\lambda_i$ in Λ (including the blocks of order one) then there exist more than one eigenvector of F that correspond to λ_i. These vectors, which are all independent, will be referred to as ' multiple vectors '. Likewise, we shall refer to the eigenvector that corresponds to an eigenvalue of Λ and possesses only one Jordan block as a ' single eigenvector '.

Using the transformation $U\mathbf{z}_k=\mathbf{x}_k$ we obtain the following equivalent representation for (2.1)

$$\mathbf{z}_{k+1}=\Lambda\mathbf{z}_k+\mathbf{u}_k$$
$$\mathbf{y}_k=J\mathbf{z}_k+\mathbf{v}_k \tag{2.3}$$

where $\mathbf{u}_k$ is a white-noise sequence with

$$\operatorname{cov}(\mathbf{u}_k)=Q=U^{-1}G\left[\operatorname{cov}(\mathbf{w}_k)\right]G^{\mathrm{t}}U^{-\mathrm{t}} \tag{2.4}$$

and where $J=HU$ we consider the matrix

$$C=E\left\{\begin{bmatrix}\mathbf{y}_{k+1}\\ \mathbf{y}_{k+2}\\ \vdots \\ \mathbf{y}_{k+l}\end{bmatrix}\left[\mathbf{y}_k^{\mathrm{t}}\, \mathbf{y}_{k-1}^{\mathrm{t}} \ldots \mathbf{y}_{k-l+1}^{\mathrm{t}}\right]\right\} \tag{2.5}$$

where E denotes expectation. C is the predictor covariance matrix which forms a starting point in stochastic realization and its rank is equal to the minimal order of the output process representation. In the following we assume that $\mathbf{y}_k$ is a stationary process. Then

$$C=\begin{bmatrix} R_1 & \dots & R_l \\ \vdots & & \vdots \\ R_l & \dots & R_{2l-1} \end{bmatrix} \tag{2.6 a}$$

where

$$R_i=E\{\mathbf{y}_{k+i}\mathbf{y}_k^{\mathrm{t}}\}=J\Lambda^i PJ^{\mathrm{t}} \tag{2.6 b}$$

and where P satisfies the equation

$$P=\Lambda P\Lambda^{\mathrm{t}}+Q \tag{2.6 c}$$

Hence

$$C=\begin{bmatrix} J\Lambda PJ^{\mathrm{t}} & J\Lambda^2PJ^{\mathrm{t}} & \dots & J\Lambda^lPJ^{\mathrm{t}} \\ J\Lambda^2PJ^{\mathrm{t}} & J\Lambda^3PJ^{\mathrm{t}} & \dots & J\Lambda^{l+1}PJ^{\mathrm{t}} \\ \vdots & & & \\ J\Lambda^lPJ^{\mathrm{t}} & J\Lambda^{l+1}PJ^{\mathrm{t}} & \dots & J\Lambda^{2l-1}PJ^{\mathrm{t}} \end{bmatrix}$$

We decompose C as

$$C=\Psi\Theta \tag{2.7 a}$$

where

$$\Theta=[\Lambda PJ^{\mathrm{t}} \mid \Lambda^2PJ^{\mathrm{t}} \mid \dots \Lambda^lPJ^{\mathrm{t}}] \tag{2.7 b}$$

and

$$\Psi^{\mathrm{t}}=[J^{\mathrm{t}} \mid \Lambda^{\mathrm{t}}J^{\mathrm{t}} \mid \dots \Lambda^{(l-1)\mathrm{t}}J^{\mathrm{t}}] \tag{2.7 c}$$

Since the system is observable we have, for any $l>p$, that rank $\Psi=p$. Thus for $l\geqslant p$ we obtain

$$\operatorname{rank} C=\operatorname{rank}\Theta \tag{2.8}$$

Also, clearly

$$\operatorname{rank}\Theta\leqslant p \tag{2.9}$$

In the following sections we study conditions under which (2.9) holds with the equality and the strict inequality signs.

3. A case of mode cancellation

In order to motivate the analysis of the next section, which leads to necessary and sufficient conditions for mode retention, we show in this section that given the choice of the input structure there always exists an input noise covariance such that the rank of the matrix C attains its minimum possible value, i.e. the output dimension q. Since this result is given merely to demonstrate mode cancellation, we make the simplifying assumption that the eigenvalues of the system are all different. This assumption will be removed in the next section.

Theorem 3.1

Suppose that the eigenvalues of F are all different, i.e. $s=p$, and that G is invertible. Then there exists a matrix cov $(\mathbf{w}_k)$ such that

$$\operatorname{rank} C=q \tag{3.1}$$

Proof

We have

$$\operatorname{rank} \Theta=p-m \tag{3.2}$$

where m is the number of zero rows in PJ^t. Since H and, consequently, J are of full rank, m is the number of columns in P that are perpendicular to all the columns of J^t. The left annihilating subspace of J^t is clearly of dimension $p-q$. Let us denote the full-rank matrix representation of this subspace by N then

$$NJ^t=0 \tag{3.3 a}$$

We partition N as

$$N=[N_1 \,\vdots\, N_2] \tag{3.3 b}$$

$$N_1 \in R^{(p-q)\times(p-q)} \tag{3.3 c}$$

$$N_2 \in R^{(p-q)\times q} \tag{3.3 d}$$

where we assume, without loss of generality, that N_1 is not singular. We also partition Λ as

$$\Lambda=\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \tag{3.4}$$

where

$$\Lambda_1=\operatorname{diag}(\lambda_1 \dots \lambda_{p-q}), \quad \Lambda_2=\operatorname{diag}(\lambda_{p-q+1} \dots \lambda_p)$$

and

$$Q=\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^t & Q_{22} \end{bmatrix} \tag{3.5}$$

Q_{11} and Q_{22} are symmetric positive definite matrices of dimension $p-q$ and q, respectively. We denote the resulting solution of (2.6 c) by

$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^t & P_{22} \end{bmatrix} \tag{3.6}$$

and we readily find that

$$P_{ij}=\Lambda_i P_{ij} \Lambda_j+Q_{ij}, \quad i, j=1, 2 \tag{3.7}$$

Choosing an arbitrary $Q_{11}>0$ we obtain $P_{11}>0$. Thus, denoting

$$D=N_1^{-t} P_{11} N_1^{-1} \tag{3.8}$$

and choosing

$$Q_{12}=N_1^{\mathrm{t}}\, DN_2-\Lambda_1 N_1^{\mathrm{t}}\, DN_2\Lambda_2 \tag{3.9}$$

it is found from (3.7) that

$$P_{12}=N_1^{\mathrm{t}}\, DN_2 \tag{3.10}$$

and hence that the first $p-q$ rows of P

$$[P_{11} \mid P_{12}]=N_1^{\mathrm{t}}\, DN$$

are perpendicular to all the rows of J. In order to complete the choice for Q it is required to show how, given $Q_{11}>0$ and Q_{12} of (3.9), it is always possible to find Q_{22} such that $Q>0$. We consider the matrix

$$\hat{Q}_i=\left[\begin{array}{c:c} Q_{11} & Q_{12}[\mathbf{e}_1 \ldots \mathbf{e}_i] \\ \hdashline \begin{bmatrix}\mathbf{e}_1^{\mathrm{t}} \\ \vdots \\ \mathbf{e}_i^{\mathrm{t}}\end{bmatrix} Q_{12}^{\mathrm{t}} & \begin{bmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_i\end{bmatrix}\end{array}\right] \tag{3.11}$$

where $\mathbf{e}_i=[00 \ldots 010 \ldots 0]^{\mathrm{t}}$ is the ith unit vector in R^{p-q}. Using Schur's formula (Wiener 1949, p. 650) we obtain

$$\det[\hat{Q}_{i+1}]=\det[\hat{Q}_i][q_{i+1}-\mathbf{e}_{i+1}^{\mathrm{t}}\, Q_{12}^{\mathrm{t}}\, \hat{Q}_i^{-1}\, Q_{12}\mathbf{e}_{i+1}] \tag{3.12}$$

Thus, denoting $\hat{Q}_0 \triangleq Q_{11}$ and choosing

$$q_i>\mathbf{e}_i^{\mathrm{t}}\, Q_{12}^{\mathrm{t}}(\hat{Q}_{i-1})^{-1}Q_{12}\mathbf{e}_i \tag{3.13}$$

we readily obtain that $\det[\hat{Q}_i]>0$ and hence, $\hat{Q}_i>0$, $\forall i=1, \ldots, q$. Thus, it is found that any Q_{22} that satisfies

$$Q_{22}>\operatorname{diag}\{\mathbf{e}_i^{\mathrm{t}}\, Q_{12}^{\mathrm{t}}\, \hat{Q}_{i-1}^{-1}\, Q_{12}\mathbf{e}_i, \quad i=1, \ldots, q\} \tag{3.14}$$

also results in Q of (3.5) which is positive definite. To complete the proof we take

$$\operatorname{cov}(\mathbf{w}_k)=G^{-1}UQU^{\mathrm{t}}G^{-\mathrm{t}} \tag{3.15}$$

□

Example 3.1

We take $\Lambda=\operatorname{diag}(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$ and $J=[3\ \ 2\ \ 1]$ and find that

$$N=\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

hence

$$N_1=I_2, \quad N_2=\begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

Choosing, say

$$Q_{11}=\begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix}$$

we obtain from (3.7) and (3.8) that

$$D=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{bmatrix} D \begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{bmatrix}+\begin{bmatrix}4 & 0\\ 0 & 16\end{bmatrix}$$

and thus

$$D=\begin{bmatrix}\frac{16}{3} & 0\\ 0 & 18\end{bmatrix}$$

From (3.9) we get

$$Q_{12}=\begin{bmatrix}-14\\ -33\end{bmatrix}$$

so that

$$Q=\begin{bmatrix}4 & 0 & -14\\ 0 & 16 & -33\\ -14 & -33 & q_1\end{bmatrix}$$

where from (3.13) it is required that $q_1>118$. If we choose, say, $q_1=225$ we find that

$$P=\begin{bmatrix}\frac{16}{3} & 0 & -16\\ 0 & 18 & -36\\ -16 & -36 & 240\end{bmatrix}$$

and therefore that

$$PJ^{t}=\begin{bmatrix}0\\ 0\\ 120\end{bmatrix}$$

It clearly follows that rank $C=1$.

4. Conditions for mode retention and cancellation

In § 3 it was shown that for a certain class of input covariance maximal mode cancellation may occur. We now turn to examine more general conditions for mode cancellation and retention. Note in particular that the following analysis applies to systems with possibly non-distinct and multiple distinct eigenvalues.

Lemma 4.1

The matrix Θ has a maximal rank p iff the columns of JP corresponding to multiple eigenvectors in Λ are independent and the columns of JP corresponding to single eigenvectors in Λ are not zero.

Proof

The proof readily follows from the Popov–Belevitch–Hautus test for controllability (Kailath 1980, p. 135) which in our case will state that the pair (Λ, PJ^t) is controllable iff there is no eigenrow of Λ that is orthogonal to PJ^t. As Λ is already given in Jordan canonical form the eigenrows of Λ are unit vectors in R^p. The assertion of the lemma thus follows from the fact that any linear combination of multiple eigenrows corresponding to the same eigenvalue of Λ is also an eigenrow of Λ.

We consider next eqn. (2.6 *c*). It is well known (Wiener 1949, p. 663) that this equation can be written as

$$[I - \Lambda \otimes \Lambda]\boldsymbol{\chi} = \boldsymbol{\gamma} \tag{4.1 a}$$

where

$$\boldsymbol{\chi} = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_p \end{bmatrix} \tag{4.1 b}$$

$$\boldsymbol{\gamma} = \begin{bmatrix} \mathbf{q}_1 \\ \vdots \\ \mathbf{q}_p \end{bmatrix} \tag{4.1 c}$$

where $\mathbf{p}_i$ and $\mathbf{q}_i$ are the ith columns of P and Q, respectively, and $\Lambda \otimes \Lambda$ is the Kronecker product of the two matrices. In our case

$$\Lambda \otimes \Lambda = \text{diag}\,\{\boldsymbol{\Phi}_i, \quad i = 1, \ldots, n\} \tag{4.2 a}$$

where

$$\boldsymbol{\Phi}_i = \begin{cases} \lambda_i \,\text{diag}\,\{\Lambda \ldots \Lambda\} & \text{for } i \leqslant s \\ \begin{bmatrix} \lambda_i\Lambda & \Lambda & & & \\ & \ddots & \ddots & & 0 \\ & & \lambda_i\Lambda & \Lambda & \\ & 0 & & \ddots & \Lambda \\ & & & & \lambda_i\Lambda \end{bmatrix} & \text{for } i > s \end{cases} \tag{4.2 b}$$

is a $(\sigma_i p \times \sigma_i p)$ matrix. Since $\boldsymbol{\Phi}_i$ is upper triangular, $I - \Lambda \otimes \Lambda$ is also upper triangular and since the system is stable, we obtain that $I - \Lambda \otimes \Lambda$ is invertible. We thus find that

$$\boldsymbol{\chi} = (I_{p^2} - \Lambda \otimes \Lambda)^{-1}\boldsymbol{\gamma} \tag{4.3}$$

and denoting

$$P_i = [\mathbf{p}_{\tau_i+1}{}^t \ldots \mathbf{p}_{\tau_i+\sigma_i}{}^t]^t \tag{4.4 a}$$

$$Q_i = [\mathbf{q}_{\tau_i+1}{}^t \ldots \mathbf{q}_{\tau_i+\sigma_i}{}^t]^t \tag{4.4 b}$$

and

$$J_i = \operatorname{diag}\{\overbrace{J, J, \ldots, J}^{\leftarrow \sigma_i \text{ times} \rightarrow}\}, \quad i = 1, \ldots, n \tag{4.4 c}$$

where

$$\tau_i = \sum_{j=1}^{i-1} \sigma_j \tag{4.4 d}$$

we find that

$$\begin{aligned} P_i &= [I_{\sigma_i p} - \Phi_i]^{-1} Q_i \\ &= \operatorname{diag}\{\overbrace{(I_p - \lambda_i \Lambda)^{-1}, (I_p - \lambda_i \Lambda)^{-1} \ldots (I_p - \lambda_i \Lambda)^{-1}}^{\longleftarrow \sigma_i \text{ times} \longrightarrow}\} \\ &\quad \times \begin{bmatrix} I_p & T_i & T_i^2 \cdots\cdots & T_i^{\sigma_i - 1} \\ & I_p & T_i & T_i^2 \\ & & \ddots & T_i \\ 0 & & & I_p \end{bmatrix} Q_i \end{aligned} \tag{4.5}$$

where we define

$$T_i = \begin{cases} 0_p & i \leqslant s \text{ or } i > s \text{ and } \sigma_i = 1 \\ \Lambda(I - \lambda_i \Lambda)^{-1} & i > s \text{ and } \sigma_i > 1 \end{cases} \tag{4.6}$$

and

$$J_i P_i = \begin{bmatrix} J\mathbf{p}_{\tau_i + 1} \\ \vdots \\ J\mathbf{p}_{\tau_i + \sigma_i} \end{bmatrix} = A(i) Q_i \tag{4.7 a}$$

where

$$A(i) = \operatorname{diag}\{\overbrace{J(I_p - \lambda_i \Lambda)^{-1}, \ldots, J(I_p - \lambda_i \Lambda)^{-1}}^{\longleftarrow \sigma_i \text{ times} \longrightarrow}\} \begin{bmatrix} I_p & T_i \cdots & T_i^{\sigma_i - 1} \\ & \ddots & \vdots \\ & I_p & T_i \\ 0 & & \ddots \\ & & I_p \end{bmatrix} \tag{4.7 b}$$

In the case where all the eigenvalues of F are distinct ($\sigma_i = 1$, $i > s$), we find from the last equation that

$$A(i) = \operatorname{diag}\{J(I_p - \lambda_i \Lambda)^{-1} \ldots J(I_p - \lambda_i \Lambda)^{-1}\} \tag{4.8}$$

The following result provides a geometrical insight into the mode cancellation mechanism.

Theorem 4.1

In the case where $\lambda_i \neq \lambda_j$, $i, j = 1, \ldots, n$, the matrix C has a maximal rank ρ iff the space spanned by the columns of Q corresponding to the multiple eigenvectors associated with λ_i is disjoint of the kernel $\{J(I - \lambda_i \Lambda)^{-1}\}$, $i = 1, \ldots, s$ and Q_i corresponding to the eigenvalue λ_i, $i > s$, is not perpendicular to the first q rows of $A(i)$.

Proof

In the case of $\lambda_i \neq \lambda_j$ the columns of JP corresponding to λ_i, $i = 1, \ldots, s$, are found by (4.7 *a*) where $A(i)$ is given by (4.8). Thus, the columns of $J[\mathbf{p}_{\tau_i+1} \cdots \mathbf{p}_{\tau_i+\sigma_i}]$ are independent iff the subspace span by the corresponding i columns of Q is disjoint of the kernel $\{J(I - \lambda_i \Lambda)^{-1}\}$. The columns of JP corresponding to single eigenvectors for λ_i, $i > s$, are given by

$$J\mathbf{p}_{\tau_i+1} = [A(i)]_1 Q_i$$

where $[A(i)]_1$ is the block built of the first q rows of $A(i)$. It thus follows that $J\mathbf{p}_{\tau_i+1} \neq 0$ iff Q_i is not perpendicular to the rows of $[A(i)]_1$. The theorem is thus proved using the result of Lemma 4.1. □

Example 4.1

Suppose that the system (2.1) has a single output ($q = 1$) and distinct eigenvalues. Then the fact that it is minimal implies that it has no multiple eigenvalues (i.e. $s = 0$ and $\sigma_i = 1$, $i = 1, \ldots, p$). Since it may be assumed, without loss of generality, to be in (or transformed into) the standard observable form, where F is a bottom companion matrix (Kailath 1980, p. 659), we have $H = [1 \ 0 \ldots 0]$ and the diagonalization transformation U is the Vandermonde matrix. Consequently $J = [1 \ 1 \ldots 1]$ and

$$A(i) = \left[\frac{1}{1 - \lambda_i \lambda_1} \ \frac{1}{1 - \lambda_i \lambda_2} \cdots \frac{1}{1 - \lambda_i \lambda_p} \right]$$

The matrix C is then of rank p iff the ith column of Q is not orthogonal to $A^t(i)$, $\forall i = 1, \ldots, p$. If we take, say

$$F = \Lambda = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}, \quad H = J = [1 \ 1]$$

and denote

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}$$

The maximal rank condition becomes

$$\frac{q_{11}}{1 - \frac{1}{4}} \neq \frac{-q_{12}}{1 + \frac{1}{6}}, \quad \frac{q_{12}}{1 + \frac{1}{6}} \neq \frac{-q_{22}}{1 - \frac{1}{9}}$$

or

$$q_{12} \neq \frac{-14}{9} q_{11}, \quad q_{12} \neq \frac{-21}{16} q_{22}$$

It can thus be seen that if $q=1$, the input structures

$$G=\begin{bmatrix} g_1 \\ \frac{-14}{9} g_1 \end{bmatrix}, \quad G=\begin{bmatrix} g_1 \\ \frac{-16}{21} g_1 \end{bmatrix}$$

must be ruled out if order identification is desired.

The following corollary specifies classes of matrices F and G for which a matrix C of maximal rank can be obtained.

Corollary 4.1

Given that the eigenvalues of F are all different and that G is invertible, there always exists cov $(\mathbf{w}_k)$ which yields a matrix C of maximal rank p.

Proof

It follows from the fact that G is invertible that for cov $(\mathbf{w}_k)=G^{-1}UDU^{t}G^{-t}$ the resulting matrix Q is D. If we choose D to be diagonal positive definite and if (F, H) is observable then J has no column that is identically zero and thus from (4.8) we find that none of the columns of D is perpendicular to the corresponding $A(i)$. The assertion thus follows from Theorem 4.1. □

So far we have studied conditions for mode retention and we have seen cases where mode cancellation may be encountered. We next show that for any input, output and covariance matrices there always exists a matrix F that yields a rank deficient C.

Theorem 4.2

Given any cov $(\mathbf{w}_k)$ and any full rank matrices H and G, there always exists a matrix F whose eigenvalues are all inside the unit circle such that (F, G) is controllable and rank $C<p$.

Proof

We prove the theorem by constructing the required F. We choose the transformation matrix U of (2.2 a) such that one of the columns in $Q=U^{-1}G$ cov $(\mathbf{w}_k)G^{t}U^{-t}$, say, the first one, has no zero entries and the first column in HU is not identically zero. This can always be done. One such choice is as follows :

We first choose a non-singular matrix U_1 such that

$$U_1^{-1}\, G \text{ cov } (\mathbf{w}_k)G^{t}U_1^{-t} \triangleq \{\hat{g}_{ij}\}$$

has no identically zero rows ; clearly, such a matrix always exists for a full rank G and cov $(\mathbf{w}_k)\neq 0$. It follows from the fact that $\{\hat{g}_{ij}\}$ is a covariance matrix that $\hat{g}_{ii}>0$, $i=1, \ldots, p$, since a zero on the diagonal implies an identically zero row. Assuming that there exists a vector $\mathbf{m}_h \in$ kernel (HU_1) whose first element is 1, namely, $\mathbf{m}_n{}^t \triangleq [1, m_{n,2}, \ldots, m_{n,p}]$ we define the non-singular diagonal matrix $\hat{D} = \text{diag}\{1, \hat{d}_2, \ldots, \hat{d}_p\}$ where

$$\hat{d}_i > m_{h,i} \frac{\hat{g}_{11}}{\hat{g}_{i,1}}, \quad i = 2, \ldots, p \tag{4.9}$$

and choose

$$U^{-1} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \alpha_2 & 1 & 0 & & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ \alpha_p & 0 & \cdots & \cdots & 1 \end{bmatrix} \hat{D} U_1{}^{-1}, \quad \alpha_i \neq 0, \quad i = 2, \ldots, p \tag{4.10}$$

such that

(i)
$$\alpha_i \neq -\hat{d}_i \frac{\hat{g}_{i1}}{\hat{g}_{11}}, \quad i = 2, \ldots, p$$

and

(ii)
$$\left[1 - \frac{\alpha_2}{\hat{d}_2} - \cdots - \frac{\alpha_p}{\hat{d}_p}\right]^t \notin \text{kernel}\,(HU_1)$$

Clearly such U guarantees that none of the entries in the first column of Q

$$\mathbf{q}_1 = \begin{bmatrix} q_{11} \\ q_{21} \\ \vdots \\ \vdots \\ q_{p1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & & & \bigcirc \\ \alpha_2 & 1 & \ddots & & \\ \vdots & & \ddots & 0 & \\ \vdots & & & \ddots & 0 \\ \alpha_p & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{g}_{11} \\ \hat{d}_2 \hat{g}_{21} \\ \vdots \\ \vdots \\ \hat{d}_p \hat{g}_{p1} \end{bmatrix} \tag{4.11}$$

is zero and that the first column of HU is not zero.

We next use the degrees of freedom left in selecting the parameters α_i, $i = 2, \ldots, p$, to ensure the existence of a matrix

$$\Lambda = \text{diag}\,(\lambda_i, i = 1, \ldots, p), \quad \lambda_i \neq \lambda_j, \quad i \neq j, \quad |\lambda_i| < 1, \quad i = 1, \ldots, p$$

that satisfies the requirement

$$(I - \lambda_i \Lambda)^{-1} \mathbf{q}_1 \in \text{kernel}\,(HU)$$

This requirement, if satisfied, would imply by (4.8) that $A(1)\mathbf{q}_1 = 0$ and thus by Theorem 4.1 that C is rank deficient.

We choose a particular vector in the kernel (HU)

$$\mathbf{m} \triangleq \begin{bmatrix} 1 \\ \mathbf{m}_2 \\ \vdots \\ \mathbf{m}_p \end{bmatrix} = \begin{bmatrix} 1 & & & \bigcirc \\ \alpha_2 & 1 & & \\ \alpha_3 & 0 & 1 & \\ \vdots & & & \ddots \\ \alpha_p & 0 & \cdots & 1 \end{bmatrix} \mathbf{m}_h \tag{4.12}$$

In order to obtain a solution for

$$(I - \lambda_1 \Lambda)^{-1} \mathbf{q}_1 = \beta \mathbf{m} \tag{4.13}$$

where β is a scalar, we choose α_i such that

(iii) $$m_i \neq 0, \quad i = 2, \ldots, p$$

The solution of (4.13) is given by

$$\lambda_1 \lambda_i = 1 - \frac{q_{i1}}{\beta m_i}, \quad i = 1, \ldots, p \quad \text{where } m_1 \triangleq 1$$

It readily follows from (4.9), (4.11) and (4.12) that $q_{i1}/m_i > q_{11} = \hat{g}_{11}$, $i = 2, \ldots, p$. Thus, requiring further that

(iv) $$\frac{q_{i1}}{m_i} \neq \frac{q_{j1}}{m_j}, \quad i \neq j$$

and choosing

$$\beta > \max_{i=1,\ldots,p} \left\{ \frac{q_{i1}}{m_i} \right\}$$

we obtain that $0 < \lambda_i < \lambda_1 < 1$.

In order to obtain a non-zero first entry in $\mathbf{m}$ we have assumed above that the first entry of $\mathbf{m}_h$ is not zero. If all the vectors in the kernel (HU_1) happen to be perpendicular to $[100 \ldots 00]^t$ we can always apply a renumerating permutation of the states and interchange the columns of HU_1 accordingly such that there exists a vector $\mathbf{m}_h$ in the resulting kernel (HU_1) whose first element is not zero. □

Example 4.2

We consider a third-order system with, say

$$H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad \operatorname{cov}\{\mathbf{w}_k\} = I_2$$

We find that

$$G \operatorname{cov}(\mathbf{w}_k) G^t = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

and thus $U_1 = I_3$ and $(\hat{g}_{11}\ \hat{g}_{21}\ \hat{g}_{31}) = (2\ 2\ 2)$. As $\mathbf{m}_h{}^t = [1\ 0\ -1]$ we choose, by (4.9), $\hat{D} = I_3$.

The requirements for α_2 and α_3 are found to be

(i)
$$\alpha_2,\ \alpha_3 \neq -1$$

(ii)
$$(1-\alpha_2-\alpha_3)^t \notin \text{kernel } (HU_1) = [1 \ \ 0 \ \ -1]$$

hence $\alpha_2 \neq 0$ and $\alpha_3 \neq 1$.

(iii)
$$m = \begin{bmatrix} 1 \\ \alpha_2 \\ \alpha_3 - 1 \end{bmatrix}$$

and thus requirement (iii) is redundant.

(iv)
$$\frac{2+2\alpha_2}{\alpha_2} \neq \frac{2+2\alpha_3}{\alpha_3 - 1}$$

and hence we require that $\alpha_2 > 0$, $\alpha_3 > 1$ and $\alpha_3 - 2\alpha_2 \neq 1$. Choosing, say, $\alpha_2 = 1$, $\alpha_3 = 2$ we find that

$$\beta > \max\left(\frac{2+2}{1}, \frac{2+4}{1}, 2\right)$$

If we choose say $\beta = 8$ we obtain that the solution of

$$(I - \lambda_1 \Lambda)^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is

$$\Lambda = \text{diag}\left(\frac{\sqrt{3}}{2} \ \ \frac{\sqrt{3}}{3} \ \ \frac{\sqrt{3}}{4}\right)$$

and

$$F = \frac{\sqrt{3}}{12} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 & & \bigcirc \\ & 4 & \\ \bigcirc & & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

As the choice of U is arbitrary and it is only required that one of the columns of the resulting Q will have any arbitrary non-zero entries and as the choice of β is also arbitrary, within the above given bounds, it appears that the resulting class of ' problematic ' matrices F cannot be ruled out as ' non-generic '.

5. Conclusion

This paper has investigated the relationship between the internal structure of a stochastic linear system and the order of its minimal output representation. The analysis was centred about the Hankel correlation matrix associated with the output. Necessary and sufficient conditions for cancellation of system modes in the output representation have been derived. These conditions specify the relationships between the input, the output and the internal (eigenvector–eigenvalue) structure of the system, which cause mode cancellation and retention. The numerical examples illustrate that while the mode cancellation problem may be considered non-generic for certain system structures (cf. Example 4.1) it is inherent in others (Example 4.2). Mode cancellation implies that the order of the system cannot be determined from its output and that the minimal order of an output filter and a feedback compensator would be lower than that of the given system.

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