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Model validation using mismatched filters

YORAM BARAM†

The problem of testing a given model for an observation sequence using residuals obtained from a sub-optimal filter is treated. Tests on cross-record sections of multiple data records are defined and the necessary distributions for computing critical regions and the tests' power for given model alternatives are derived.

1. Introduction

This paper is concerned with the problem of validating a linear model, using estimation residuals obtained from mismatched (suboptimal) filters. Such filters are often used in practice when, for reasons of implementation, the filter must be of lesser complexity than the model assumed for the system. The problem of testing mismatched filter residuals differs in several ways from that of testing matched filter residuals (e.g. Box and Jenkins 1970, Mehra and Peschon 1971). While matched filter residuals are uncorrelated, mismatched filter residuals are generally correlated. In addition, mismatched filter residuals cannot be made stationary via normalization by the associated covariance matrix as matched filter residuals. The problem of testing mismatched filter residuals is solved in this paper by using multiple data records. Such records are often available from repeated experiments or from multiple system operations, and are particularly useful when the data obtained from a single experiment is statistically insufficient.

Statistical analysis on multiple data records is particularly attractive because cross-record residual samples are, normally, independent and identically distributed. It is only necessary that the driving and measurement noise sequences corresponding to different experiments be independent and normally distributed. Statistical inference from multiple data records has been previously suggested by Goodrich and Caines (1979), who considered linear system identification, and by Baram (1980) who considered model validation from matched filter residuals.

Mean, covariance and correlation tests on cross-record sections of the data are defined. The residuals are projected onto the principal directions of the covariance and correlation matrices in order to efficiently transform the matrix-valued information in the corresponding sample matrices into scalar test statistics. This defines another point of departure from the matched filter case as the correlation matrix for matched filter residuals is the identity matrix. The test statistics are written as quadratic forms in normal variables so that the distribution theory of such forms can be used to compute critical regions for the tests and the tests' power for specified model alternatives.

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† Department of Electrical Engineering, Tel Aviv University, Ramat Aviv, Tel Aviv, Israel.

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2. Test statistics

Consider a family of observation records $y_{i,j} \in R^p$, $i=1, \ldots, M$, $j=1, \ldots, N$, where i is the record index and j is the sampling (time) index. It is assumed that the statistical model of the observations is the same for all the data records. Let M^0 denote the model assumed for the observations (the model to be tested) and let M^1 denote another model for the observations used for estimation purposes (the model in the estimator). We shall also have a third model M^2, representing the true system generating the observations when the tested model M^1 does not hold. Let E^l denote expectation with respect to the probability measure induced by the model M^l for $l=0, 1, 2$. The observation residuals are given by

$$r_{i,j}{}^l = y_{i,j}{}^l - E^1\{y_{i,j} | y_{i,1}, \ldots, y_{i,j-1}\} \quad l=1, 2 \tag{2.1}$$

when the observations are generated by M^0 and M^2, respectively. Let us denote

$$m_j{}^l = E^l\{r_{i,j}\} \quad l=0, 2 \tag{2.2}$$

$$P_j{}^l = E^l\{(r_{i,j} - m_j)(r_{i,j} - m_j)^{\mathrm{T}}\} \quad l=0, 2 \tag{2.3}$$

and

$$C^l(j, k) = E^l\{(r_{i,k} - m_j)(r_{i,k} - m_k)^{\mathrm{T}}\} \quad l=0, 2 \tag{2.4}$$

where $E^l\{r_{i,j}\} = E^l\{r_{i,j}{}^l\}$, etc.

The computation of $m_j{}^l$, $P_j{}^l$ and $C^l(j, k)$ for linear models is described in the appendix. In the sequel the index l will be dropped when the analysis applies equally to $l=0$ and $l=2$.

For a cross-record section j the sample mean is computed as

$$\bar{r}_j = \frac{1}{M} \sum_{i=1}^{M} r_{i,j} \tag{2.5}$$

The mean test will employ the statistic

$$\|\bar{r}_j\| = \bar{r}_j{}^{\mathrm{T}} \bar{r}_j \tag{2.6}$$

Let $V(j, k)$ denote the matrix whose columns are the principal directions of $C^0(j, k)$. Note that $C(j, k)$ may be singular. In general $V(j, k)$ may be obtained by singular valued decomposition of $C(j, k)$, (e.g., Golub and Reinach 1970). The residuals are projected onto the γth principal direction of $C(j, k)$ by the operation

$$\rho_{i,j}{}^{(\gamma)}(j, k) = a^{(\gamma)\mathrm{T}} V^{\mathrm{T}}(j, k)[r_{i,j} - \bar{r}_j] \tag{2.7}$$

where $a^{(\gamma)}$ is a vector whose α'th component is given by

$$a_\alpha{}^{(\gamma)} = \begin{cases} 1 & \alpha = \gamma \\ 0 & \alpha \neq \gamma \end{cases} \tag{2.8}$$

The correlation test statistic may be defined as the sample correlation of the projected residuals

$$c^{(\gamma)}(j, k) = \frac{1}{M} \sum_{i=1}^{M} \rho_{i,j}{}^{(\gamma)}(j, k) \rho_{i,k}{}^{(\gamma)}(j, k) \tag{2.9}$$

Note that when the tested model is correct

$$\lim_{M\to\infty} c^{(\gamma)}(j,k)=\lambda^{(\gamma)} \tag{2.10}$$

where $\lambda^{(\gamma)}$ denotes the γth eigenvalue of $C(j,k)$. When the eigenvalues are ordered by their magnitudes, the sample correlation $c^{\gamma}(j,k)$ can be expected to be more accurate (i.e., less affected by round-off errors) for $\gamma=1$ than for $\gamma=2$ etc. This also translates the multi-variate information contained in the sample correlation matrix into a smaller number of scalars, ordered by their significance. Another correlation test statistic may be defined as

$$c(j,k)=\sum_{\gamma=1}^{p} c^{(\gamma)}(j,k) \tag{2.11}$$

Note that

$$c(j,k)=\frac{1}{M}\sum_{i=1}^{M}(r_{i,j}-\bar{r}_j)^{\mathrm{T}}(r_{i,k}-\bar{r}_k)=\operatorname{tr}\bar{C}(j,k) \tag{2.12}$$

where $\bar{C}(j,k)$ is the residual sample correlation matrix. This is due to the fact that the trace of $\bar{C}(j,k)$ is preserved under the similarity transformation $V^{\mathrm{T}}(j,k)\bar{C}(j,k)V(j,k)$.

Statistics for testing the instantaneous covariance are obtained as special cases of the above statistics, by taking $j=k$. Let V_j be the matrix of principal directions of P_j. Then projecting the residuals on the γth principal direction

$$\rho_{i,j}{}^{(\gamma)}=a^{(\gamma)}V_j(r_{i,j}-\bar{r}_j) \tag{2.13}$$

we have the covariance test statistic

$$c_j{}^{(\gamma)}=\frac{1}{M}\sum_{i=1}^{M}\rho_{i,j}{}^{(\gamma)} \tag{2.14}$$

When the tested model is correct we have

$$\lim_{M\to\infty} c_j{}^{(\gamma)}=\lambda^{(\gamma)} \tag{2.15}$$

where $\lambda^{(\gamma)}$ is the γth eigenvalue of P_j. The covariance test statistics may also be defined as

$$c_j=\sum_{\gamma=1}^{p} c_j{}^{(\gamma)} \tag{2.16}$$

As in the case of the sample correlation we have

$$c_j=\frac{1}{M}\sum_{i=1}^{M}(r_{i,j}-\bar{r}_j)^{\mathrm{T}}(r_{i,j}-\bar{r}_j)=\operatorname{tr}\bar{P}_j \tag{2.17}$$

where $\bar{P}_j$ is the sample covariance matrix.

3. Test statistics distribution

The test statistics presented in the previous section will now be written as quadratic forms in normal random variables. Well established techniques for computing the distributions of such forms can then be used to derive the necessary distributions.

For a cross-record section j let us define

$$\phi_j = (r_{1,j}^{\mathrm{T}}, r_{2,j}^{\mathrm{T}}, \ldots, r_{M\,j}^{\mathrm{T}})^{\mathrm{T}} \tag{3.1}$$

The mean test statistic (2.4) can be written as

$$\|\bar{r}_j\| = \phi_j^{\mathrm{T}} A \phi_j \tag{3.2}$$

where

$$A = \frac{1}{M^2}\begin{bmatrix} I_n & I_n & \ldots & I_n \\ I_n & I_n & \ldots & I_n \\ \vdots & \vdots & & \vdots \\ I_n & I_n & \ldots & I_n \end{bmatrix} \tag{3.3}$$

where I_n is the n-dimensional identity matrix. Let us denote

$$\phi(j, k) = (\phi_j^{\mathrm{T}}, \phi_k^{\mathrm{T}})^{\mathrm{T}} \tag{3.4}$$

The correlation test statistics (2.9) and (2.11) can be written as

$$c^{(\gamma)}(j, k) = \phi(j, k)^{\mathrm{T}} B^{(\gamma)}(j, k) \phi(j, k) \tag{3.5}$$

and

$$c(j, k) = \phi(j, k)^{\mathrm{T}} B \phi(j, k) \tag{3.6}$$

where

$$B^{(\gamma)}(j, k) = D^{(\gamma)}(j, k) B D^{(\gamma)\mathrm{T}}(j, k) \tag{3.7}$$

$$D^{(\gamma)}(j, k) = \begin{bmatrix} V(j, k)a^{(\gamma)} & 0_{p\times 1} & \ldots & 0_{p\times 1} \\ 0_{p\times 1} & V(j, k)a^{(\gamma)} & \ldots & 0_{p\times 1} \\ \vdots & \vdots & & \vdots \\ 0_{p\times 1} & 0_{p\times 1} & \ldots & V(j, k)a^{(\gamma)} \end{bmatrix} \tag{3.8}$$

$$B = \frac{1}{2M}\begin{bmatrix} 0_{M\times M} & J \\ J & 0_{M\times M} \end{bmatrix} \tag{3.9}$$

and

$$J = I_M - \frac{1}{M}\begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & & & \\ 1 & 1 & \ldots & 1 \end{bmatrix} \tag{3.10}$$

The covariance test statistics (2.14) and (2.16) can be written as

$$c_j^{(\gamma)} = \phi_j^{\mathrm{T}} G_j^{(\gamma)} \phi_j \tag{3.11}$$

and

$$c_j = \phi_j^{\mathrm{T}} G \phi_j \tag{3.12}$$

where

$$G_j^{(\gamma)} = D_j^{(\gamma)} G D_j^{(\gamma)\mathrm{T}} \tag{3.13}$$

$$D_j^{(\gamma)} = \begin{bmatrix} V_j a^{(\gamma)} & 0_{p\times 1} & \cdots & 0_{p\times 1} \\ 0_{p\times 1} & V_j a^{(\gamma)} & \cdots & 0_{p\times 1} \\ \vdots & \vdots & & \\ 0_{p\times 1} & 0_{p\times 1} & \cdots & V_j a^{(\gamma)} \end{bmatrix} \tag{3.14}$$

and

$$G = \frac{1}{M} J \tag{3.15}$$

Equations (3.2), (3.5), (3.6), (3.11) and (3.12) specify, respectively, the mean, correlation and covariance test statistics as quadratic forms in the normal vectors ϕ_j and $\phi(j, k)$. Such forms have been treated extensively in the statistical literature (e.g. Graybill 1976). It has been shown (Imhoff, 1961) that the characteristic function for the distribution of a quadratic form $x^{\mathrm{T}} A x$, where $x \sim N\ (m, \Sigma)$ and A is symmetric, is given by

$$\phi(t) = \prod_{k=1}^{n} (1 - 2i\lambda_k t)^{-1/2h_k} \exp\left\{ i \sum_{k=1}^{n} \frac{\delta_k^2 \lambda_k t}{1 - 2_i \lambda_k t} \right\} \tag{3.16}$$

where λ_k is the kth eigenvalue of $\Sigma^{1/2} A \Sigma^{1/2}$, h_k is its multiplicity and δ_k is given by

$$\delta_k = p_k^{\mathrm{T}} \Sigma^{-1/2} m \tag{3.17}$$

where p_k is the kth eigenvector of $\Sigma^{1/2} A \Sigma^{1/2}$. A numerical technique for computing the distribution from the characteristic function has been suggested by Imhoff (1961), along with an approximation technique.

In order to define the critical regions for the tests, it is necessary to determine the test statistics distributions for the mismatched combination (M^0, M^1) (i.e. the distributions induced by the residuals of the filter matched to M^1 when the model M^0 is correct). In order to find the tests' power for a specified alternative model M^2, it is necessary to find the distributions corresponding to the mismatched combination (M^2, M^1). Since the test statistics are now given as quadratic forms in ϕ_j and $\phi(j, k)$, it remains to specify the mean and covariance of these normal vectors for the two mismatched combinations. The mean values are given by

$$E\phi_j^l = m_j^l = (m_{1,j}^{l\mathrm{T}}, m_{2,j}^{l\mathrm{T}}, \ldots, m_{M,j}^{l\mathrm{T}})^{\mathrm{T}} \quad l = 0, 2 \tag{3.18}$$

$$E\phi^l(j, k) = (m_j^{l\mathrm{T}}, m_k^{l\mathrm{T}})^{\mathrm{T}} \quad l = 0, 2 \tag{3.19}$$

The covariance of ϕ_j^l is given by the blocks

$$(\mathrm{cov}\ \phi_j^l)_{\alpha,\beta} = \begin{cases} P_{j,\alpha}^l & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases} \quad l = 0, 2 \tag{3.20}$$

The covariance of $\phi^l(j, k)$ is given by

$$\operatorname{cov} \phi^l(j,k) = \begin{bmatrix} U_j^{\,l} & U_{j,k}^{\,l} \\ U_{j,k}^{\,l\mathrm{T}} & U_k^{\,l} \end{bmatrix} \quad l=0,\,2 \tag{3.21}$$

where

$$U_j^{\,l} = \operatorname{cov} \phi_j^{\,l} \quad l=0,\,2 \tag{3.22}$$

and

$$(U_{j,k}^{\,l})_{\alpha,\beta} = \begin{cases} C_\alpha^{\,l}(j,k) & \alpha=\beta \\ 0 & \alpha \neq \beta \end{cases} \quad l=0,\,2 \tag{3.23}$$

4. Conclusion

The problem of model validation using residuals obtained from a mismatched filter differs from the classical model validation problem in that the residuals are non-zero mean, non-stationary and correlated even when the model to be validated is correct. In this paper we have presented a method for solving this problem, using multiple data records. Mean, covariance and correlation tests on cross-record sections of the data were defined. Projecting the residuals onto the principal directions of the covariance and correlation matrices provides a meaningful way of transforming the matrix-valued information into scalar statistics. The test statistics distributions were derived for computing critical regions for the tests, and for computing the tests' power for specified model alternatives.

Appendix

The distribution of mismatched filter residuals for linear state-space models

Let M^0 denote a model for the observation sequence (y_i), $i=1, 2, \ldots, N$ and let M^1 be the model used in the filter. We are interested in the probability distribution of the filter's residuals. Here we derived the distribution for the mismatched combination (M^0, M^1). The distribution for the alternative (M^2, M^1) required for test power computation purposes can be derived in the same manner, replacing the index 0 by 2. Consider the linear state-space model

$$x_{j+1} = F_j x_j + G_j w_j \tag{A 1}$$

$$y_j = H_j x_j + v_j \tag{A 2}$$

where x_0 is normally distributed with

$$E\{x_0\} = m_0 \tag{A 3}$$

$$E\{(x_0 - m_0)(x_0 - m_0)^{\mathrm{T}}\} = \Psi_0 \tag{A 4}$$

and where w_j and v_j are uncorrelated and mutually uncorrelated normal sequences with

$$E\{w_j\} = m_{w,j} \tag{A 5}$$

$$E\{(w_j - m_{w,j})(w_j - m_{w,j})^{\mathrm{T}}\} = Q_j \tag{A 6}$$

and

$$E\{v_j\} = m_{v,j} \tag{A 7}$$

$$E\{(v_i - m_{v,j})(v_j - m_{v,j})^{\mathrm{T}}\} = R_j \tag{A 8}$$

the mismatched filter residuals r_j^0 are generated by the equations

$$\tilde{x}_{j+1} = \tilde{F}_j \tilde{x}_j + \tilde{G}_j \tilde{w}_j \tag{A 9}$$

$$r_j^0 = \tilde{H}_j \tilde{x}_j + v_j^0 \tag{A 10}$$

where

$$\tilde{x}_j = \begin{bmatrix} x_j^0 \\ \hat{x}_j^1 \end{bmatrix} \tag{A 11}$$

$$\tilde{w}_j = \begin{bmatrix} w_j^0 \\ w_j^1 \\ v_j^0 \end{bmatrix} \tag{A 12}$$

$$\tilde{F}_j = \begin{bmatrix} F_j^0 & \\ F_j^1 K_j^1 H_j^0 & F_j^1(I - K_j^1 H_j^1) \end{bmatrix} \tag{A 13}$$

$$\tilde{G}_j = \begin{bmatrix} G_j^0 & 0 & 0 \\ 0 & G_j^1 & F_j^1 K_j^1 \end{bmatrix} \tag{A 14}$$

$$\tilde{H}_j = [H_j^0 - H_j^1] \tag{A 15}$$

K_j^0 is obtained from

$$K_j^0 = \Sigma_j^0 H_j^{0\mathrm{T}} (H_j^0 \Sigma_j^0 H_j^{0\mathrm{T}} + R_j^0)^{-1} \tag{A 17}$$

where Σ_j^0 is obtained recursively from

$$\Sigma_j^0 = F_j^0(I - K_j^0 H_j)\Sigma_j^0 F_j^{0\mathrm{T}} + Q_j^0 \tag{A 18}$$

initialized by

$$\Sigma_0^0 = \psi_0 \tag{A 19}$$

Also let

$$\tilde{Q}_j = \begin{bmatrix} Q_j^0 & 0 & 0 \\ 0 & Q_j^1 & 0 \\ 0 & 0 & R_j^0 \end{bmatrix} \tag{A 20}$$

$$\tilde{x}_0 = \begin{bmatrix} m_0^0 \\ m_0^1 \end{bmatrix} \tag{A 21}$$

$$\tilde{u}_j = \begin{bmatrix} m_{w,j}^0 \\ m_{w,j}^1 \\ m_{v,j}^0 \end{bmatrix} \tag{A 22}$$

The residual mean is then

$$m_j = \tilde{H}_j \Theta(j, 0)\tilde{x}_0 + \tilde{H}_j \sum_{k=1}^{j-1} \Theta(j, k)\tilde{G}_k \tilde{u}_k \tag{A 23}$$

where $\Theta(j, k)$ is the transition matrix corresponding to $\tilde{F}_j$. To obtain the residual covariance function let

$$\tilde{\Psi}_j = \text{cov}\ (\tilde{x}_j) \tag{A 24}$$

$\tilde{\Psi}$ is computed by the recursion

$$\tilde{\Psi}_j = \tilde{F}_j\, \tilde{\Psi}_j\, \tilde{F}_j^{\mathrm{T}} + \tilde{G}_j \tilde{Q}_j \tilde{G}_j^{\mathrm{T}} \tag{A 25}$$

initialized at

$$\tilde{\Psi}_0 = \begin{bmatrix} \tilde{\Psi}_0{}^0 & 0 \\ 0 & 0 \end{bmatrix} \tag{A 26}$$

Then the residual covariance function is given by

$$C(j, k) = \begin{cases} P_j & j = k \\ \tilde{H}_j \tilde{\Psi}_j \Theta^{\mathrm{T}}(j, k)\tilde{H}_k^{\mathrm{T}} & j < k \\ \tilde{H}_j \Theta(k, j)\tilde{\Psi}_k \tilde{H}_k^{\mathrm{T}} & j > k \end{cases} \tag{A 27}$$

where

$$P_j = \tilde{H}_j \tilde{\Psi}_j \tilde{H}_j^{\mathrm{T}} + R_j^0 \tag{A 28}$$

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